

ORDINARY VARIETIES AND THE COMPARISON BETWEEN MULTIPLIER IDEALS AND TEST IDEALS II

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ABSTRACT. We consider the following conjecture: if X is a smooth n -dimensional projective variety in characteristic zero, then there is a dense set of reductions X_s to positive characteristic such that the action of the Frobenius morphism on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective. We also consider the conjecture relating the multiplier ideals of an ideal \mathfrak{a} on a nonsingular variety in characteristic zero, and the test ideals of the reductions of \mathfrak{a} to positive characteristic. We prove that the latter conjecture implies the former one.

1. INTRODUCTION

This note is motivated by the joint paper with V. Srinivas [MS], aimed at understanding the following conjecture relating invariants of singularities in characteristic zero with corresponding invariants in positive characteristic. For a discussion of the notions involved, see below.

Conjecture 1.1. *Let Y be a smooth, irreducible variety over an algebraically closed field k of characteristic zero, and \mathfrak{a} a nonzero ideal on Y . Given any model Y_A and \mathfrak{a}_A for Y and \mathfrak{a} over a subring A of k , finitely generated over \mathbf{Z} , there is a dense set of closed points $S \subset \operatorname{Spec} A$ such that*

$$(1) \quad \mathcal{J}(Y, \mathfrak{a}^\lambda)_s = \tau(Y_s, \mathfrak{a}_s^\lambda)$$

for every $\lambda \in \mathbf{R}_{\geq 0}$ and every $s \in S$.

In the conjecture, we denote by Y_s the fiber of Y_A over $s \in S$, and \mathfrak{a}_s is the ideal on Y_s induced by \mathfrak{a}_A . The ideals $\mathcal{J}(Y, \mathfrak{a}^\lambda)$ are the multiplier ideals of \mathfrak{a} . These are fundamental invariants of the singularities of \mathfrak{a} , that have seen a lot of recent applications due to their appearance in vanishing theorems (see [Laz, Chapter 9]). The ideals $\tau(Y_s, \mathfrak{a}_s^\lambda)$ are the (generalized) test ideals of Hara and Yoshida [HY], defined in positive characteristic using the Frobenius morphism. The above conjecture asserts therefore that for a dense set of closed points, we have the equality between the test ideals of \mathfrak{a} and the reductions of the multiplier ideals of \mathfrak{a} for *all* exponents. We note that it is shown in [HY] that if $\lambda \in \mathbf{R}_{\geq 0}$ is fixed, then the equality in (1) holds for every s in an open subset of the closed points in $\operatorname{Spec} A$.

The following conjecture was proposed in [MS].

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Conjecture 1.2. *Let X be a smooth, irreducible n -dimensional projective variety defined over an algebraically closed field k of characteristic zero. If X_A is a model of X defined over a subring A of k , finitely generated over \mathbf{Z} , then there is a dense set of closed points $S \subseteq \operatorname{Spec} A$ such that the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective for every $s \in S$.*

It is expected, in fact, that there is a set S as in Conjecture 1.2 such that X_s is ordinary in the sense of Bloch and Kato [BK] for every $s \in S$. In particular, this would imply that the action of the Frobenius on each cohomology group $H^i(X_s, \mathcal{O}_{X_s})$ is bijective (see [MS, Remark 5.1]). The main result of [MS] is that Conjecture 1.2 implies Conjecture 1.1. In this note we show that the converse is true:

Theorem 1.3. *If Conjecture 1.1 holds, then so does Conjecture 1.2.*

The following is an outline of the proof. Given a variety X as in Conjecture 1.2, we embed it in a projective space \mathbf{P}_k^N such that $r := N - n \geq n + 1$, and the ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_N]$ defining X is generated by quadrics. In this case it is easy to compute the multiplier ideals $\mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda)$ for $\lambda < r$, and in particular we see that $(x_0, \dots, x_N)^{2r-N-1} \subseteq \mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda)$ for every $\lambda < r$. It follows from a general property of multiplier ideals that if g_1, \dots, g_r are general linear combinations of a system of generators of \mathfrak{a} , and if $h = g_1 \cdots g_r$, then $\mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda) = \mathcal{J}(\mathbf{A}_k^{N+1}, h^{\lambda/r})$ for every $\lambda < r$. In this case, Conjecture 1.1 implies that for a dense set of closed points $s \in \operatorname{Spec} A$, the ideal $(x_0, \dots, x_N)^{2r-N-1}$ is contained in $\tau(\mathbf{A}_{k(s)}^{N+1}, h_s^\mu)$ for every $\mu < 1$. Using some basic properties of test ideals, we deduce that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective, where $D_s \subset \mathbf{P}_{k(s)}^N$ is the hypersurface defined by h_s . We show that this in turn implies the bijectivity of the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$, hence proves the theorem.

2. PROOF OF THE MAIN RESULT

We start by recalling the definition of multiplier ideals and test ideals. Suppose first that Y is a smooth, irreducible variety over an algebraically closed field k of characteristic zero, and \mathfrak{a} is a nonzero ideal on Y . A *log resolution* of \mathfrak{a} is a projective, birational morphism $\pi: W \rightarrow Y$, with W smooth, such that $\mathfrak{a} \cdot \mathcal{O}_W$ is the ideal of a divisor D on W , with $D + K_{W/Y}$ having simple normal crossings (as usual, $K_{W/Y}$ denotes the relative canonical divisor of W over Y). With this notation, for every $\lambda \in \mathbf{R}_{\geq 0}$ we have

$$(2) \quad \mathcal{J}(Y, \mathfrak{a}^\lambda) = \pi_* \mathcal{O}_W(K_{W/Y} - \lfloor \lambda D \rfloor).$$

Recall that if $E = \sum_i a_i E_i$ is a divisor with \mathbf{R} -coefficients, then $\lfloor E \rfloor = \sum_i \lfloor a_i \rfloor E_i$, where $\lfloor t \rfloor$ is the largest integer $\leq t$. It is a well-known fact that the above definition is independent of the choice of log resolution. For this and other basic facts about multiplier ideals, see [Laz, Chapter 9].

Suppose now that $Y = \operatorname{Spec} R$ is an affine smooth, irreducible scheme of finite type over a perfect field L of positive characteristic p (in the case of interest for us, L will be a finite field). Under these assumptions, the test ideals admit the following simple description that we will use, see [BMS2]. Recall that for an ideal J and for $e \geq 1$, one denotes by $J^{[p^e]}$ the ideal $(h^{p^e} \mid h \in J)$. One can show that given an ideal \mathfrak{b} in R , there

is a unique smallest ideal J such that $\mathfrak{b} \subseteq J^{[p^e]}$; this ideal is denoted by $\mathfrak{b}^{[1/p^e]}$. Suppose now that \mathfrak{a} is an ideal in R and $\lambda \in \mathbf{R}_{\geq 0}$. One can show that for every $e \geq 1$ we have the inclusion

$$(\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq (\mathfrak{a}^{\lceil \lambda p^{e+1} \rceil})^{[1/p^{e+1}]},$$

where $\lceil t \rceil$ denotes the smallest integer $\geq t$. Since R is Noetherian, it follows that $(\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]}$ is constant for $e \gg 0$. This is the test ideal $\tau(Y, \mathfrak{a}^\lambda)$. For details and a discussion of basic properties of test ideals in this setting, we refer to [BMS2]. For a comparison of general properties of multiplier ideals and test ideals, see [HY] and [MY].

If \mathfrak{a} is an ideal in the polynomial ring $k[x_0, \dots, x_N]$, where k is a field of characteristic zero, a *model* of \mathfrak{a} over a subring A of k , finitely generated over \mathbf{Z} , is an ideal \mathfrak{a}_A in $A[x_0, \dots, x_N]$ such that $\mathfrak{a}_A \cdot k[x_0, \dots, x_N] = \mathfrak{a}$. We can obtain such a model by simply taking A to contain all the coefficients of a finite system of generators of \mathfrak{a} . Of course, we may always replace A by a larger ring with the same properties; in particular, we may replace A by a localization A_a at a nonzero element $a \in A$. If $s \in \text{Spec } A$ and if \mathfrak{a}_A is a model of \mathfrak{a} , then we obtain a corresponding ideal \mathfrak{a}_s in $k(s)[x_0, \dots, x_N]$. Note that if s is a closed point, then the residue field $k(s)$ is a finite field.

Suppose now that $X \subseteq \mathbf{P}_k^N$ is a projective subscheme defined by the homogeneous ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_N]$. If $\mathfrak{a}_A \subseteq A[x_0, \dots, x_N]$ is a model of \mathfrak{a} over A , which we may assume homogeneous, then the subscheme X_A of \mathbf{P}_A^N defined by \mathfrak{a}_A is a model of X over A . If $s \in \text{Spec } A$, then the subscheme $X_s \subseteq \mathbf{P}_{k(s)}^N$ is defined by the ideal \mathfrak{a}_s . We refer to [MS, §2.2] for some of the standard facts about reduction to positive characteristic. We note that given \mathfrak{a} as above, we may consider simultaneously all the reductions $\mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda)_s$ for all $\lambda \in \mathbf{R}_{\geq 0}$. This is due to the fact that for bounded λ we only have to deal with finitely many ideals, while for $\lambda \gg 0$, the multiplier ideals are determined by the lower ones via a Skoda-type theorem (see [MS, §3.2] for details).

We can now give the proof of our main result stated in Introduction.

Proof of Theorem 1.3. Let X be a smooth, irreducible n -dimensional projective variety over an algebraically closed field k of characteristic zero, with $n \geq 1$. It is clear that the assertion we need is independent of the model X_A that we choose. Consider a closed embedding $X \hookrightarrow \mathbf{P}_k^N$. After replacing this by a composition with a d -uple Veronese embedding, for $d \gg 0$, we may assume that the saturated ideal $\mathfrak{a} \subset R = k[x_0, \dots, x_N]$ defining X is generated by homogeneous polynomials of degree two (see [ERT, Proposition 5]). Furthermore, we may clearly assume that $r := N - n \geq n + 1$. Under these assumptions, it is easy to determine the multiplier ideals of \mathfrak{a} of exponent $< r$.

Lemma 2.1. *With the above notation, if $\mathfrak{m} = (x_0, \dots, x_N)$, then*

$$\mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda) = \begin{cases} R, & \text{if } 0 \leq \lambda < \frac{N+1}{2}; \\ \mathfrak{m}^{\lfloor 2\lambda \rfloor - N}, & \text{if } \frac{N+1}{2} \leq \lambda < r. \end{cases}$$

Proof. Let us fix $\lambda \in \mathbf{R}_{\geq 0}$, with $\lambda < r$. We denote by Z the subscheme of \mathbf{A}_k^{N+1} defined by \mathfrak{a} . Let $\varphi: W \rightarrow \mathbf{A}_k^{N+1}$ be the blow-up of the origin, with exceptional divisor E . Since \mathfrak{a} is generated by homogeneous polynomials of degree two, it follows that $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}_W(-2E) \cdot \tilde{\mathfrak{a}}$,

where $\tilde{\mathfrak{a}}$ is the ideal defining the strict transform \tilde{Z} of Z on W . We have $K_{W/\mathbf{A}_k^{N+1}} = NE$, hence the change of variable formula for multiplier ideals (see [Laz, Theorem 9.2.33]) implies

$$(3) \quad \mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda) = \varphi_* (\mathcal{J}(W, (\mathfrak{a} \cdot \mathcal{O}_W)^\lambda) \otimes \mathcal{O}_W(NE)).$$

It is clear that \tilde{Z} is nonsingular over $\mathbf{A}_k^{N+1} \setminus \{0\}$. Since $\tilde{Z} \cap E \subseteq E \simeq \mathbf{P}^N$ is isomorphic to the scheme X , hence it is nonsingular, it follows that \tilde{Z} is nonsingular, and \tilde{Z} and E have simple normal crossings. Let $\psi: \tilde{W} \rightarrow W$ be the blow-up of W along \tilde{Z} , with exceptional divisor T , and let \tilde{E} be the strict transform of E . Note that \tilde{W} is nonsingular, and $\tilde{E} + T$ has simple normal crossings. We have $K_{\tilde{W}/W} = (r-1)T$ and $\mathfrak{a} \cdot \mathcal{O}_{\tilde{W}} = \mathcal{O}_{\tilde{W}}(-2\tilde{E} - T)$. Therefore ψ is a log resolution of $\mathfrak{a} \cdot \mathcal{O}_W$, and by definition we have

$$(4) \quad \mathcal{J}(W, (\mathfrak{a} \cdot \mathcal{O}_W)^\lambda) = \psi_*(\mathcal{O}_{\tilde{W}}(-([\lambda] - r + 1)T - [2\lambda]\tilde{E}) = \mathcal{O}_W(-[2\lambda]E)$$

(recall that $\lambda < r$). The formula in the lemma follows from (3), (4), and the fact that $\varphi_*(\mathcal{O}_W(-iE)) = \mathfrak{m}^i$ for every $i \in \mathbf{Z}_{\geq 0}$. \square

Let f_1, \dots, f_m be a system of generators of \mathfrak{a} , with each f_i homogeneous of degree two. We fix g_1, \dots, g_r general linear combinations of the f_i with coefficients in k , and put $h = g_1 \cdots g_r$. In this case, we have

$$(5) \quad \mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda) = \mathcal{J}(\mathbf{A}_k^{N+1}, h^{\lambda/r})$$

for every $\lambda < r$ (see [Laz, Proposition 9.2.28]).

Suppose now that \mathfrak{a}_A and h_A are homogeneous models of \mathfrak{a} , and respectively h , over A . Let $X_A, D_A \subset \mathbf{P}_A^N$ be the projective schemes defined by \mathfrak{a}_A and h_A , respectively. Note that g_1, \dots, g_r being general linear combinations of the f_i , the subscheme $V(g_1, \dots, g_r) \subset \mathbf{P}_k^N$ has pure codimension r . Therefore we may assume that for every $s \in \text{Spec } A$, the scheme $V((g_1)_s, \dots, (g_r)_s)$ has pure codimension r in $\mathbf{P}_{k(s)}^N$. We need to show that given models as above, there is a dense set of closed points $S \subset \text{Spec } A$ such that the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective for every $s \in S$. The next lemma shows that in fact, it is enough to find S as above such that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective for all $s \in S$.

Lemma 2.2. *Let L be a finite field, and D_1, \dots, D_r hypersurfaces in $\mathbf{P}^N = \mathbf{P}_L^N$, with $r \leq N$, such that the intersection scheme $Y = D_1 \cap \dots \cap D_r$ has pure codimension r in \mathbf{P}^N . If the Frobenius action on $H^{N-1}(D, \mathcal{O}_D)$ is bijective, where $D = \sum_{i=1}^r D_i$, then for every closed subscheme X of Y , the Frobenius action on $H^{N-r}(X, \mathcal{O}_X)$ is bijective.*

Proof. If $r = N$, then X is zero-dimensional, and the Frobenius action on $\Gamma(X, \mathcal{O}_X)$ is bijective since L is perfect. Therefore from now on we may assume that $r \leq N - 1$.

For every subset $J \subseteq \{1, \dots, r\}$, let $D_J = \bigcap_{j \in J} D_j$. By assumption, Y is a complete intersection, hence there is an exact complex

$$\mathcal{C}^\bullet: 0 \rightarrow \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \dots \xrightarrow{d^{r-1}} \mathcal{C}^r \rightarrow 0,$$

where $\mathcal{C}^0 = \mathcal{O}_D$, and $\mathcal{C}^m = \bigoplus_{|J|=m} \mathcal{O}_{D_J}$ for $m \geq 1$. Note that we have a morphism of complexes $\mathcal{C}^\bullet \rightarrow F_*(\mathcal{C}^\bullet)$, where F is the absolute Frobenius morphism on X . It follows that if we break-up \mathcal{C}^\bullet into short exact sequences, the maps in the corresponding long exact sequences for cohomology are compatible with the Frobenius action.

Let $\mathcal{M}^i = \text{Im}(d^i)$, hence $\mathcal{M}^0 \simeq \mathcal{C}^0 = \mathcal{O}_D$ and $\mathcal{M}^{r-1} = \mathcal{C}^r = \mathcal{O}_Y$. Since each D_J is a complete intersection in \mathbf{P}^N , it follows that $H^i(D_J, \mathcal{O}_{D_J}) = 0$ for every i with $1 \leq i < \dim(D_J) = N - |J|$. We deduce that for every i with $0 \leq i \leq r-2$, the short exact sequence

$$0 \rightarrow \mathcal{M}^i \rightarrow \mathcal{C}^{i+1} \rightarrow \mathcal{M}^{i+1} \rightarrow 0$$

gives an exact sequence

$$0 = H^{N-i-2}(\mathbf{P}^N, \mathcal{C}^{i+1}) \rightarrow H^{N-i-2}(\mathbf{P}^N, \mathcal{M}^{i+1}) \rightarrow H^{N-i-1}(\mathbf{P}^N, \mathcal{M}^i).$$

Therefore we have a sequence of injective maps

$$H^{N-r}(Y, \mathcal{O}_Y) \hookrightarrow H^{N-r+1}(\mathbf{P}^N, \mathcal{M}^{r-2}) \hookrightarrow \dots \hookrightarrow H^{N-2}(\mathbf{P}^N, \mathcal{M}^1) \hookrightarrow H^{N-1}(D, \mathcal{O}_D),$$

compatible with the Frobenius action. Since this action is bijective on $H^{N-1}(D, \mathcal{O}_D)$ by hypothesis, it follows that it is bijective also on $H^{N-r}(Y, \mathcal{O}_Y)$ (see, for example, [MS, Lemma 2.4]).

On the other hand, since $\dim(Y) = N - r$, the surjection $\mathcal{O}_Y \rightarrow \mathcal{O}_X$ induces a surjection $H^{N-r}(Y, \mathcal{O}_Y) \rightarrow H^{N-r}(X, \mathcal{O}_X)$, compatible with the Frobenius action. As we have seen, the Frobenius action is bijective on $H^{N-r}(Y, \mathcal{O}_Y)$, hence on every quotient (see [MS, Lemma 2.4]). This completes the proof of the lemma. \square

Returning to the proof of Theorem 1.3, we see that it is enough to show that there is a dense set of closed points $S \subset \text{Spec } A$ such that Frobenius acts bijectively on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ for $s \in S$. We assume that Conjecture 1.1 holds, hence there is a dense set of closed points $S \subset \text{Spec } A$ such that $\tau(\mathbf{A}_{k(s)}^{N+1}, h_s^\lambda) = \mathcal{J}(\mathbf{A}_k^{N+1}, h^\lambda)_s$ for every $\lambda \in \mathbf{R}_{\geq 0}$ and every $s \in S$. In particular, it follows from Lemma 2.1 and (5) that $(x_0, \dots, x_N)^{2r-N-1} \subseteq \tau(\mathbf{A}_{k(s)}^{N+1}, h_s^\lambda)$ for every $\lambda < 1$. Since $\deg(h_s) = 2r \geq (N+1)$, Proposition 2.3 below implies that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective for all $s \in S$. As we have seen, this completes the proof of Theorem 1.3. \square

Proposition 2.3. *Let L be a perfect field of characteristic $p > 0$, and $h \in R = L[x_0, \dots, x_N]$ a homogeneous polynomial of degree $d \geq N+1$, with $N \geq 2$. If $(x_0, \dots, x_N)^{d-N-1} \subseteq \tau(\mathbf{A}_L^{N+1}, h^{1-\frac{1}{p}})$, then the Frobenius action on $H^{N-1}(D, \mathcal{O}_D)$ is bijective, where $D \subset \mathbf{P}_L^N$ is the hypersurface defined by h .*

Proof. In the case $d = N+1$, this is a reformulation of a well-known fact due to Fedder [Fe]. We follow the argument from [MTW, Proposition 2.16], that extends to our more general setting. It is enough to show that the Frobenius action on $H^{N-1}(D, \mathcal{O}_D)$ is injective (see [MS, §2.1]).

Note first that $\tau(\mathbf{A}_L^{N+1}, h^{1-\frac{1}{p}}) = (h^{p-1})^{[1/p]}$ (see [BMS1, Lemma 2.1]), hence by assumption $\mathfrak{m}^{d-N-1} \subseteq (h^{p-1})^{[1/p]}$, where $\mathfrak{m} = (x_0, \dots, x_N)$. It is convenient to use the interpretation of the ideal $(h^{p-1})^{[1/p]}$ in terms of local cohomology. Let $E = H_{\mathfrak{m}}^{N+1}(R)$.

Recall that this is a graded R -module, carrying a natural action of the Frobenius, that we denote by F_E . There is an isomorphism

$$E \simeq R_{x_0 \cdots x_N} / \sum_{i=0}^N R_{x_0 \cdots \widehat{x_i} \cdots x_N}.$$

Via this isomorphism, F_E is induced by the Frobenius morphism on $R_{x_0 \cdots x_N}$.

The annihilator of $(h^{p-1})^{[1/p]}$ in E is equal to $\text{Ker}(h^{p-1}F_E)$ (see, for example, [BMS2, §2.3]). Therefore we have

$$(6) \quad \text{Ker}(h^{p-1}F_E) \subseteq \text{Ann}_E(\mathfrak{m}^{d-N-1}) = \bigoplus_{i \geq -d+1} E_i.$$

On the other hand, the exact sequence

$$0 \rightarrow R(-d) \xrightarrow{h} R \rightarrow R/(h) \rightarrow 0$$

induces an isomorphism

$$H_{\mathfrak{m}}^N(R/(h)) \simeq \{u \in E \mid hu = 0\}(-d),$$

such that the Frobenius action on $H_{\mathfrak{m}}^N(R/(h))$ is given by $h^{p-1}F_E$. Since $H^{N-1}(D, \mathcal{O}_D) \simeq H_{\mathfrak{m}}^N(R/(h))_0 \hookrightarrow E_{-d}$, (6) implies that the Frobenius action is injective on $H^{N-1}(D, \mathcal{O}_D)$. This completes the proof of the proposition. \square

Remark 2.4. In the proof of Theorem 1.3 we only used the inclusion “ \subseteq ” in Conjecture 1.1. However, this is the interesting inclusion: the reverse one is known, see [HY] or [MS, Proposition 4.2]. It is more interesting that we only used Conjecture 1.1 when $Y = \mathbf{A}_k^{N+1}$, \mathfrak{a} is principal and homogeneous, and $\lambda = 1 - \frac{1}{p}$. By combining Theorem 1.3 with the main result in [MS], we see that in order to prove Conjecture 1.1 in general, it is enough to consider the case when $Y = \mathbf{A}_k^n$, $\mathfrak{a} = (f)$ is principal and homogeneous, and show the following: if $\mathfrak{b} = \mathcal{J}(Y, \mathfrak{a}^{1-\varepsilon})$ for $0 < \varepsilon \ll 1$, and if $f_A \in A[x_1, \dots, x_n]$ is a model for f , then there is a dense set of closed points $S \subset \text{Spec } A$ such that

$$\mathfrak{b}_s \subseteq (f_s^{p-1})^{[1/p]}$$

for every $s \in S$, where $p = \text{char}(k(s))$.

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